

FACTORIALS THROUGH THE LENS OF COMPLEX ANALYSIS
MATH 202B
MIDTERM
NADIR HAJOUJI

1. INTRODUCTION: THE FACTORIAL FUNCTION

The factorial function $! : \mathbb{N} \leftarrow \mathbb{N}$, given by $n! = \prod_{k=1}^n k$ (where the empty product is taken to be 1 by convention), is quite remarkable. Its definition suggests that it lives entirely within the realms of the natural numbers, and yet it is the key building block of all of the transcendental functions. Since one can produce transcendence from the factorials of natural numbers, who knows what beautiful theories might arise if we had reasonable definitions for $\frac{1}{2}!$ or $\pi!$, say. Such theories are beyond the scope of this paper, and the author is truly dismayed that time will not permit those ideas, or get to discuss the relationship between the Gamma function and the zeta function. The recent paper by Bhargava (see [3]) shows some of the other directions one could take the study of the factorial.

A few general comments should be made at this point. First, let us note that it is conventional, when one is studying the factorial in this type of general context, to study the “translated” factorial, i.e. the map $n \mapsto (n-1)!$. One reason to do this is that this map can be characterized as $n \mapsto \frac{\prod_{k=1}^n k}{n}$, and the 0 case no longer appears as a degenerate case. Second, with this convention in place, it clearly suffices to check that $f(1) = 1$ and:

$$f(n) = (n-1)f(n-1) \quad (\dagger)$$

to show that $f(n) = (n-1)!$ for all n . The functional equation \dagger will come up a lot.

2. GAMMA FUNCTION

Our first task is to generalize the factorial function to some open neighborhood in \mathbb{C} . By Weierstrass, we know that given any open set Ω , $\Delta \subset \Omega$ a discrete set and $c_\lambda \in \mathbb{C}$ for each $\lambda \in \Delta$, we can find a function f that is analytic on Ω and for which $f(\lambda) = c_\lambda$ for all $\lambda \in \Delta$. We could take our Δ to be the positive integers, and define $c_\lambda = (\lambda - 1)!$.

To construct f , we need a function g that has vanishing of first order at λ_n . From the work we did with the sine function, we know that we can simply take:

$$g(z) = \prod_1^\infty \left(1 - \frac{z}{n}\right) e^{z/n}$$

The next step in the process is usually to find functions $h_n(z)$ which will cause:

$$g(z) \sum_{n=1}^\infty \frac{(n+1)!}{g'(n)(z-n)} + h_n(z)$$

to converge. Remarkably, this will not be necessary. Observe that:

$$\pi z g(z) g(-z) = \pi z \prod_{n \neq 0 \in \mathbb{Z}} \left(1 - \frac{z}{n}\right) e^{z/n} = \sin \pi z$$

$$\implies z g(z) g(-z) = \frac{\sin \pi z}{\pi} \quad (\star)$$

Let us turn our attention to the auxiliary function in that display:

$$G(z) = g(-z) = \prod_1^\infty \left(1 + \frac{z}{n}\right) e^{-z/n}$$

It's clear that $G(z)$ has simple zeros at all of the negative integers, and $G(z-1)$ has simple zeros at all of the nonpositive integers. Let $h(z) = \frac{G(z-1)}{zG(z)}$. Then $h(z)$ is analytic on \mathbb{C} , and $h(z) \neq 0$ for all $z \in \mathbb{C}$, so we can take the log of h . Let:

$$\gamma(z) = \text{Log}(h(z)) = \left(\sum_{n=1}^\infty \log \left(1 + \frac{z-1}{n}\right) - \frac{z-1}{n} \right) - \left(\log z + \sum_{n=1}^\infty \log \left(1 + \frac{z}{n}\right) - \frac{z}{n} \right)$$

Compute:

$$\begin{aligned}
\gamma'(z) &= \left(\sum_{n=1}^{\infty} \frac{\frac{1}{n}}{1 + \frac{z-1}{n}} - \frac{1}{n} \right) - \left(\frac{1}{z} + \sum_{n=1}^{\infty} \frac{\frac{1}{n}}{1 + \frac{z}{n}} - \frac{1}{n} \right) \\
&= \left(\sum_{n=1}^{\infty} \frac{1}{n + (z-1)} - \frac{1}{n} \right) - \left(\frac{1}{z} + \sum_{n=1}^{\infty} \frac{1}{n+z} - \frac{1}{n} \right) \\
&= \left(\frac{1}{z} - 1 \right) + \left(\sum_{n=1}^{\infty} \frac{1}{n+z} - \frac{1}{n+1} \right) - \left(\frac{1}{z} + \sum_{n=1}^{\infty} \frac{1}{n+z} - \frac{1}{n} \right) \\
&= -1 + \sum_{n=1}^{\infty} \frac{1}{n} - \frac{1}{n+1}
\end{aligned}$$

I claim that $\sum_{n=1}^N \frac{1}{n} - \frac{1}{n+1} = \frac{n}{n+1}$. Indeed, $\sum_{n=1}^1 \frac{1}{n} - \frac{1}{n+1} = \frac{1}{2} = \frac{1}{1+1}$, and:

$$\begin{aligned}
\sum_{n=1}^N \frac{1}{n} - \frac{1}{n+1} = \frac{N}{N+1} &\implies \sum_{n=1}^{N+1} \frac{1}{n} - \frac{1}{n+1} = \frac{N}{N+1} + \left(\frac{1}{N+1} - \frac{1}{N+2} \right) \\
&= \frac{(N+1)(N+2) - (N+1)}{(N+1)(N+2)} \\
&= \frac{N+1}{N+2}
\end{aligned}$$

so the claim holds by induction. Thus $\sum_{n=1}^{\infty} = 1$ so $\gamma'(z) = -1 + 1 = 0$, so $\gamma(z)$ is constant!

Since $\gamma(z)$ is constant, $h(z)$ is constant, so we may drop the (z) from their expression. We have shown that $\frac{G(z-1)}{zG(z)} = h$ is constant, so we have a functional equation $G(z-1) = hzG(z)$ which looks something like we want. If we consider the function $H(z) = hG(z) = e^{\gamma z}G(z)$ instead, we see that:

$$(z+1)H(z+1) = (z+1)e^{\gamma(z+1)}G(z+1) = e^{\gamma z}e^{\gamma}G(z+1) = e^{\gamma z}G(z) = H(z)$$

so H satisfies an even better functional equation. However, the “order” needs to be reversed in order obtain \dagger . It turns out that the reciprocal of $zH(z)$ does the job, and in fact that reciprocal is the famous Γ function, expressed as an infinite product, in the style of Weierstrass:

$$\boxed{\Gamma(z) = \frac{e^{-\gamma z}}{z} \prod_1^{\infty} \left(1 + \frac{z}{n}\right)^{-1} e^{z/n}}$$

Proposition 2.1. For every positive integer n , $\Gamma(n) = (n - 1)!$

Proof. First, observe that:

$$\Gamma(z)z = \frac{z}{zH(z)} = \frac{1}{H(z)} = \frac{1}{zH(z+1)} = \Gamma(z+1) \quad (\dagger)$$

so Γ satisfies \dagger .

It now suffices to show that $\Gamma(1) = 1$ to prove that $\Gamma(z) = (z - 1)!$ for all positive integers z . We have:

$$G(0) = \prod_1^{\infty} \left(1 + \frac{0}{n}\right) e^{-0/n} = 1 = e^{\gamma}G(1)$$

so:

$$\Gamma(1) = \frac{1}{1 \cdot H(1)} = \frac{1}{e^{\gamma}G(1)} = \frac{1}{1} = 1$$

Thus $\Gamma(n) = (n - 1)!$ for all n . □

Furthermore, Γ satisfies a second functional equation, which resembles \star somewhat.

Proposition 2.2.

$$\Gamma(z)\Gamma(1 - z) = \frac{\pi}{\sin \pi z}$$

Proof. Replacing z by $-z$ in \dagger , we obtain:

$$\Gamma(-z)(-z) = G(1 + (-z)) = G(1 - z) \quad (\dagger')$$

Thus:

$$\begin{aligned} \Gamma(z)\Gamma(1 - z) &= \Gamma(z)(-z)\Gamma(-z) = \frac{-z}{zH(z)H(-z)} \\ &= \frac{1}{ze^{-\gamma z}G(z)e^{\gamma z}G(-z)} = \frac{1}{zG(z)G(-z)} = \frac{\pi}{\sin \pi z} \quad (\star') \end{aligned}$$

□

3. THE COTANGENT FUNCTION

We now have a much better understanding of $n!$, having generalized it analytically to a domain that includes all but countably many complex numbers. Furthermore, the journey was pleasant: we did not need to fuss about the genus of g when we constructed g , and we did not even need to use our interpolation formula. Simply replacing $g(x)$ by $\frac{1}{g(-x)}$ produced the function we wanted. It is therefore natural to suspect, or at least hope, that we might see some more beautiful mathematics if we ask more questions about the factorial function.

One thing that we still don't know is approximately how big $n!$ is for an arbitrary integer n without actually computing it. In other words, we would like bounds on $\Gamma(z)$, or $n!$, so that we can estimate expressions involving $n!$ when they invariably arise in computations. Ideally, we would like an elementary function that grows asymptotically like $n!$, and whose error can be controlled.

One can prove, from first principles, that:

$$\frac{n^{n-1}}{e^{n-1}} < (n-1)! < \frac{n^n}{e^{n-1}}$$

so there is hope. We know we are looking for a function of the form $(\frac{n}{e})^{n-1}\epsilon(n)$, where $1 < \epsilon(n) < n$.

It will be easier to study the growth of $\log \Gamma(z)$. The principal branch is a natural choice for \log , the poles of Γ lie on the negative real axis anyways. We can restrict our attention to the right half plane $\operatorname{Re} z \geq 1$ to help keep things under control.

Now, to understand how something grows, we study its derivatives. Compute that:

$$\begin{aligned} \log \Gamma(z) &= -\gamma z - \log z + \sum_1^{\infty} \frac{z}{n} - \log \left(1 + \frac{z}{n}\right) \\ \implies \frac{d}{dz} \log \Gamma(z) &= -\gamma - \frac{1}{z} + \sum_1^{\infty} \frac{1}{n} - \frac{\frac{1}{n}}{1 + \frac{z}{n}} = -\gamma - \frac{1}{z} + \sum_1^{\infty} \frac{1}{n} - \frac{1}{n+z} \\ \implies \frac{d^2}{dz^2} \log \Gamma(z) &= \frac{1}{z^2} + \sum_1^{\infty} \frac{1}{(n+z)^2} = \boxed{\sum_0^{\infty} \frac{1}{(n+z)^2}} \end{aligned}$$

The final expression seems particularly tidy, so let us see what we can do with it. We are looking for a different way to express Γ , so perhaps using the notion of integral representations might help us. Specifically, we would like to find a function that has residues at each of these points.

To that end, we take a brief detour to study the cotangent function. Recall that:

$$\cot \xi = \frac{\cos \xi}{\sin \xi} = \frac{e^{i\xi} + e^{-i\xi}}{2} \cdot \frac{2i}{e^{i\xi} - e^{-i\xi}} = i \frac{e^{i\xi} + e^{-i\xi}}{e^{-i\xi} - e^{i\xi}} = i \frac{e^{2i\xi} + 1}{e^{2i\xi} - 1}$$

Since sine and cosine have no roots in common, the poles of \cot are exactly the zeros of \sin , with the same multiplicity. Consequently, $\cot(\pi\zeta)$ has a simple pole at $\zeta = n$ for every integer n , so the function:

$$\Phi(\zeta) = \frac{\pi \cot \pi\zeta}{(z + \zeta)^2}$$

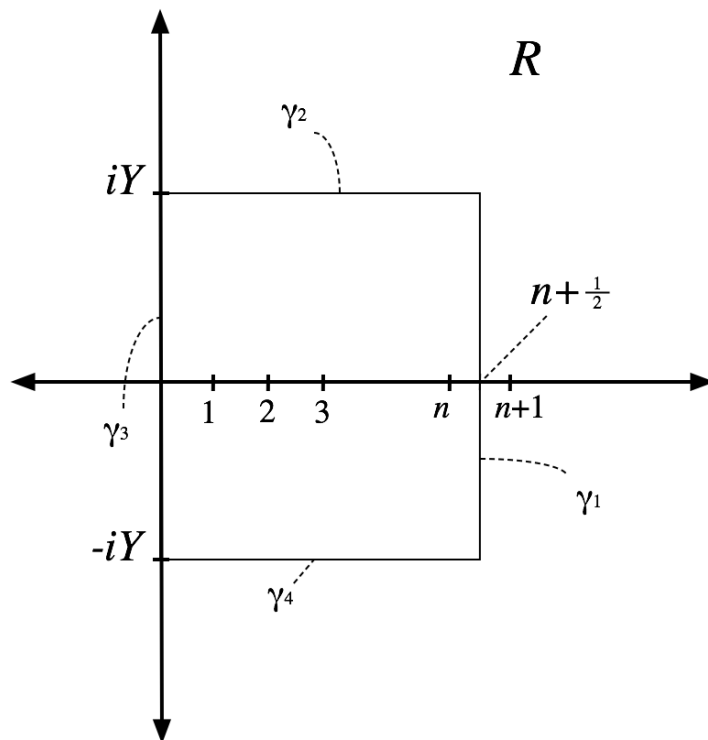
should do the trick, as long as ζ stays in the right half plane so that we do not introduce any new poles.

Note that, indeed:

$$\text{Res}_n(\Phi(\zeta)) = \lim_{\zeta \rightarrow n} \frac{\pi \cos(\pi\zeta)}{(z + \zeta)^2} \cdot \frac{\zeta - n}{\sin(\pi\zeta)} \Big|_n = \frac{\pi \cos(\pi\zeta)}{(z + \zeta)^2} \cdot \frac{1}{[\sin(\pi\zeta)]'_n} \Big|_n = \frac{1}{(z + n)^2}$$

so $\Phi(\zeta)$ has the desired residues at the desired points.

We will integrate Φ around the rectangle R shown below:



and then take the limit as $n, Y \rightarrow \infty$. The result will be a representation of $(\log \Gamma(z))''$ as an analytic function of z . We integrate twice, and compute the constants of integration, and then we will have a new representation of $\Gamma(z)$.

Let us verify that the set-up is, indeed, appropriate. We want to use the fact that $(\log \Gamma(z))'' = \sum_{\zeta \in R} \text{Res}_{\zeta}(\Phi)$ to obtain a new representation via the residue theorem, but we have a simple pole at $0 \in \partial R$. Fortunately, we can use Sohotsku-Plemelj-Privalov to proceed anyways: we simply take the principal value of the integral and add half of the residue at 0. Consequently, we see that this method is, indeed, sensible, and that:

$$\text{pr.v.} \frac{1}{2\pi i} \int_R \Phi(\zeta) d\zeta + \frac{\text{Res}_0(\Phi)}{2} = \sum_{\nu=0}^n \frac{1}{(z + \nu)^2}$$

The residue of $\Phi(\zeta)$ at 0 is $\frac{1}{z^2}$ so:

$$\text{pr.v.} \frac{1}{2\pi i} \int_R \Phi(\zeta) d\zeta = -\frac{1}{2z^2} + \sum_{\nu=0}^n \frac{1}{(z + \nu)^2} \quad (\Delta)$$

Compute that:

$$\cot(x + iy) = i \frac{e^{2ix} \cdot e^{-2y} + 1}{e^{2ix} \cdot e^{-2y} - 1}$$

Consequently, as $y \rightarrow \infty$, $e^{-2y} \rightarrow 0$ so:

$$\lim_{y \rightarrow \infty} |\cot(x + iy)| = \left| \frac{e^{2ix} \cdot 0 + 1}{e^{2ix} \cdot 0 - 1} \right| = -1$$

Similarly, as $y \rightarrow -\infty$, $e^{-2y} \rightarrow \infty$, so obviously $e^{2ix} e^{-2y} - 1 \sim e^{2ix} e^{-2y} + 1$ so:

$$\lim_{y \rightarrow -\infty} |\cot(x + iy)| = 1$$

Hence:

$$\lim_{y \rightarrow \pm\infty} \left| \Phi\left(\left(n + \frac{1}{2}\right) + iy\right) \right| = \lim_{y \rightarrow \pm\infty} \left| \frac{\cot(\pi(n + \frac{1}{2}) + iy)}{\left(\left(n + \frac{1}{2}\right) + iy + z\right)^2} \right| = \frac{1}{\lim_{y \rightarrow \pm\infty} \left| \left(\left(n + \frac{1}{2}\right) + iy + z\right) \right|^2} = 0$$

$$\implies \lim_{Y \rightarrow \infty} \left| \int_{\gamma_2} \Phi(\zeta) d\zeta \right| \leq \lim_{Y \rightarrow \infty} \int_{n+\frac{1}{2}}^0 \left| \frac{\pi \cot(\pi(\xi + iY))}{(\xi + iY + z)^2} \right| d\xi = 0$$

$$\implies \lim_{Y \rightarrow \infty} \left| \int_{\gamma_4} \Phi(\zeta) d\zeta \right| \leq \lim_{Y \rightarrow \infty} \int_0^{n+\frac{1}{2}} \left| \frac{\pi \cot(\pi(\xi - iY))}{(\xi - iY + z)^2} \right| d\xi = 0$$

We will get rid of the integral along γ_1 in a similar fashion. Compute that:

$$\cot\left(\pi\left(n + \frac{1}{2} + iy\right)\right) = i \frac{e^{(2n+1)\pi i} \cdot e^{-2\pi y} + 1}{e^{(2n+1)\pi i} \cdot e^{-2\pi y} - 1} = i \frac{-e^{-2\pi y} + 1}{-e^{-2\pi y} - 1} =_{\text{call}} c(y)$$

Thus, the numerator of $\Phi(\pi(n + \frac{1}{2}) + it)$ does not change as $n \rightarrow \infty$ (in \mathbb{Z}), but the denominator goes to ∞ , so again we have:

$$\begin{aligned} \lim_{n \rightarrow \infty} \left| \int_{\gamma_1} \Phi(\zeta) d\zeta \right| &\leq \lim_{n \rightarrow \infty} \int_{-Y}^Y \left| \frac{\pi \cot(\pi(n + \frac{1}{2} + i\eta))}{(n + \frac{1}{2} + i\eta)^2 + z^2} \right| d\eta \\ &= \lim_{n \rightarrow \infty} \int_{-Y}^Y \left| \frac{\pi c(\eta)}{(n + \frac{1}{2} + i\eta)^2 + z^2} \right| d\eta = 0 \end{aligned}$$

Finally, we get to the interesting segment: γ_3 . There is a residue at $0 \in \gamma_3$, so we compute the principal value of the integral. We manipulate the integral first:

$$\begin{aligned} \int_{\gamma_3} \Phi(\zeta) d\zeta &= \int_Y^{-Y} \frac{\pi \cot(\pi(i\eta))}{(i\eta + z)^2} d\eta = - \int_{-Y}^Y \frac{\pi \cot(\pi(i\eta))}{(i\eta + z)^2} d\eta \\ &= \lim_{\delta \rightarrow 0} - \int_{-Y}^{-\delta} \frac{\pi \coth(\pi\eta)}{(i\eta + z)^2} dt - \int_{\delta}^Y \frac{\pi \coth(\pi\eta)}{(i\eta + z)^2} dt \\ &= \lim_{\delta \rightarrow 0} - \int_{\delta}^Y \frac{\pi \coth(\pi(-\eta))}{(i(-\eta) + z)^2} d(-\eta) - \int_{\delta}^Y \frac{\pi \coth(\pi\eta)}{(i\eta + z)^2} dt \\ &= \lim_{\delta \rightarrow 0} - \int_{\delta}^Y \pi \coth(\pi\eta) \left(\frac{1}{(i\eta + z)^2} - \frac{1}{(i(-\eta) + z)^2} \right) d\eta \\ &= \lim_{\delta \rightarrow 0} - \int_{\delta}^Y \pi \coth(\pi\eta) \left(\frac{(-\eta^2 - 2iz\eta + z^2) - (-\eta^2 + 2iz\eta + z^2)}{(i\eta + z)^2(i(-\eta) + z)^2} \right) d\eta \\ &= \lim_{\delta \rightarrow 0} \int_{\delta}^Y \frac{4\pi i\eta z \coth(\pi\eta)}{(\eta^2 + z^2)^2} d\eta \end{aligned}$$

We can rewrite the display as:

$$\int_{\gamma_3} \Phi(\zeta) d\zeta = \int_0^Y \frac{4\pi i\eta z \coth(\pi\eta)}{(\eta^2 + z^2)^2} d\eta$$

so combining these results with Δ , we obtain the representation:

$$\sum_{\nu=0}^{\infty} \frac{1}{(z + \nu)^2} - \frac{1}{2z^2} = \frac{1}{2\pi i} \int_0^{\infty} \frac{4\pi i\eta z \coth(\pi\eta)}{(\eta^2 + z^2)^2} d\eta$$

which is more succinctly expressed as:

$$\boxed{\frac{d^2}{dz^2} \log \Gamma(z) = \frac{1}{2z^2} + \int_0^{\infty} \frac{2\eta z \coth(\pi\eta)}{(\eta^2 + z^2)^2} d\eta} \quad (\Delta')$$

4. STIRLING'S FORMULA

By integrating the RHS of (Δ') with respect to z twice, we will have a new expression for $\log \Gamma(z)$, which we can easily transform into a representation of $\Gamma(z)$. Compute that:

$$\coth \pi \eta = \frac{e^{\pi \eta} + e^{-\pi \eta}}{e^{\pi \eta} - e^{-\pi \eta}} = 1 + \left(\frac{e^{\pi \eta} + e^{-\pi \eta}}{e^{\pi \eta} - e^{-\pi \eta}} - 1 \right) = 1 + \frac{e^{\pi \eta} + e^{-\pi \eta} - e^{\pi \eta} + e^{-\pi \eta}}{e^{\pi \eta} - e^{-\pi \eta}} = 1 + \frac{2}{e^{2\pi \eta} - 1}$$

so we have:

$$\begin{aligned} \frac{1}{2z^2} + \int_0^\infty \coth \pi \eta \cdot \frac{2\eta z}{(\eta^2 + z^2)^2} d\eta &= \frac{1}{2z^2} + \int_0^\infty \left(1 + \frac{2}{e^{2\pi \eta} - 1} \right) \cdot \frac{2\eta z}{(\eta^2 + z^2)^2} d\eta \\ &= \frac{1}{2z^2} + \int_0^\infty \frac{d\eta}{e^{2\pi \eta} - 1} \cdot \frac{4\eta z}{(\eta^2 + z^2)^2} + \int_0^\infty \frac{2\eta z}{(\eta^2 + z^2)^2} d\eta \\ &= \frac{1}{2z^2} + \int_0^\infty \frac{d\eta}{e^{2\pi \eta} - 1} \cdot \frac{4\eta z}{(\eta^2 + z^2)^2} + \left(\frac{-z}{z^2 + \eta^2} \right)_0^\infty \\ &= \frac{1}{z} + \frac{1}{2z^2} + \int_0^\infty \frac{d\eta}{e^{2\pi \eta} - 1} \cdot \frac{4\eta z}{(\eta^2 + z^2)^2} \end{aligned}$$

The integral in the final display converges well enough that we can apply Fubini's Theorem to show:

$$\begin{aligned} \int \frac{d^2}{dz^2} \log \Gamma(z) dz &= \int \left(\frac{1}{z} + \frac{1}{2z^2} + \int_0^\infty \frac{d\eta}{e^{2\pi \eta} - 1} \cdot \frac{4\eta z}{(\eta^2 + z^2)^2} \right) dz \\ &= \log(z) - \frac{1}{2z} + \int \int_0^\infty \frac{1}{e^{2\pi \eta} - 1} \cdot \frac{4\eta z}{(\eta^2 + z^2)^2} d\eta dz \\ &= \log(z) - \frac{1}{2z} + \int_0^\infty \frac{2}{e^{2\pi \eta} - 1} \int \frac{2\eta z}{(\eta^2 + z^2)^2} dz d\eta \\ &= C + \log(z) - \frac{1}{2z} - \int_0^\infty \frac{2\eta}{(e^{2\pi \eta} - 1)(\eta^2 + z^2)} d\eta \end{aligned}$$

We need to integrate once more, but we should proceed with caution. If we simply repeat what we did above, we will find ourselves staring at the integral $\int \frac{1}{z^2 + \eta^2}$ and wondering how to make sense of the multivalued nature of \arctan . Instead, let us manipulate the integral we just obtained before integrating the entire expression with respect to z . With $u(\eta) = \frac{\eta}{\eta^2 + z^2}$ and $v(\eta) = \log(1 - e^{-2\pi \eta})$, we have:

$$v'(\eta) = \frac{(1 - e^{-2\pi \eta})'}{1 - e^{-2\pi \eta}} = \frac{-(-2\pi)e^{-2\pi \eta}}{1 - e^{-2\pi \eta}} = \frac{2\pi}{e^{2\pi \eta} - 1}$$

so:

$$\begin{aligned}
\int_0^\infty \frac{2\eta}{(e^{2\pi\eta} - 1)(\eta^2 + z^2)} d\eta &= \frac{1}{\pi} \int_0^\infty u(\eta)v'(\eta) d\eta \\
&= \frac{1}{\pi} \left(u(\eta)v(\eta)|_0^\infty - \int_0^\infty u'(\eta)v(\eta) d\eta \right) \\
&= \frac{1}{\pi} \left(\left(\lim_{t \rightarrow \infty} \frac{t \log(1 - e^{-2\pi t})}{t^2 + z^2} - \left(\frac{0}{0^2 + z^2} \right) \right) - \int_0^\infty u'(\eta)v(\eta) d\eta \right) \\
&= -\frac{1}{\pi} \int_0^\infty \left(\frac{\eta}{\eta^2 + z^2} \right)' \log(1 - e^{-2\pi\eta}) d\eta \\
&= -\frac{1}{\pi} \int_0^\infty - \left(\frac{(\eta^2 + z^2) - \eta(2\eta)}{(\eta^2 + z^2)^2} \right) \log(1 - e^{-2\pi\eta}) d\eta \\
&= -\frac{1}{\pi} \int_0^\infty \left(\frac{\eta^2 - z^2}{(\eta^2 + z^2)^2} \right) \log(1 - e^{-2\pi\eta}) d\eta
\end{aligned}$$

Thus we have:

$$\frac{d}{dz} \log \Gamma(z) = C + \log(z) - \frac{1}{2z} + \frac{1}{\pi} \int_0^\infty \left(\frac{\eta^2 - z^2}{(\eta^2 + z^2)^2} \right) \log(1 - e^{-2\pi\eta}) d\eta$$

and now the integral is perfectly harmless.

Let us integrate with respect to z once more to obtain our new representation of $\log \Gamma(z)$:

$$\begin{aligned}
\log \Gamma(z) &= C' + Cz + (z \log(z) - z) - \frac{1}{2} \log(z) + \frac{1}{\pi} \int \int_0^\infty \left(\frac{\eta^2 - z^2}{(\eta^2 + z^2)^2} \log(1 - e^{-2\pi\eta}) d\eta \right) dz \\
&= C' + (C - 1)z + \left(z - \frac{1}{2} \right) \log(z) + \frac{1}{\pi} \int_0^\infty \log(1 - e^{-2\pi\eta}) \int \left(\frac{\eta^2 - z^2}{(\eta^2 + z^2)^2} dz \right) d\eta
\end{aligned}$$

For convenience, we replace C by $C - 1$. We proceed to evaluate the integral:

$$\begin{aligned}
\int \frac{\eta^2 - z^2}{(\eta^2 + z^2)^2} dz &= \frac{z}{\eta^2 + z^2} \\
\implies \log \Gamma(z) &= C' + Cz + \left(z - \frac{1}{2} \right) \log(z) + \frac{1}{\pi} \int_0^\infty \log(1 - e^{-2\pi\eta}) \frac{z}{\eta^2 + z^2} d\eta
\end{aligned}$$

Now we will need to determine C, C' . We do so by studying:

$$J(z) = \frac{1}{\pi} \int_0^\infty \log(1 - e^{-2\pi\eta}) \frac{z}{\eta^2 + z^2} d\eta$$

We split $J(z)$ up into two parts:

$$J_1(z) = \frac{1}{\pi} \int_0^{\frac{|z|}{2}} \log(1 - e^{-2\pi\eta}) \frac{z}{\eta^2 + z^2} d\eta \quad J_2(z) = \frac{1}{\pi} \int_{\frac{|z|}{2}}^{\infty} \log(1 - e^{-2\pi\eta}) \frac{z}{\eta^2 + z^2} d\eta$$

Now, if $0 \leq \eta \leq \frac{|z|}{2}$, by the reverse triangle inequality, we have:

$$|\eta^2 + z^2| = |z^2 - (-\eta)^2| \geq ||z|^2 - |\eta|^2| = |z|^2 - \eta^2 \geq |z|^2 - \left(\frac{|z|}{2}\right)^2 = \frac{3|z|^2}{4}$$

so taking reciprocals, and then multiplying by $|z|$, we get:

$$\frac{1}{|\eta^2 + z^2|} \leq \frac{4}{3|z|^2} \implies \frac{|z|}{|\eta^2 + z^2|} \leq \frac{4}{3|z|}$$

Consequently:

$$\begin{aligned} |J_1(z)| &= \left| \frac{1}{\pi} \int_0^{\frac{|z|}{2}} \log(1 - e^{-2\pi\eta}) \frac{z}{\eta^2 + z^2} d\eta \right| \\ &\leq \frac{1}{\pi} \int_0^{\frac{|z|}{2}} |\log(1 - e^{-2\pi\eta})| \left| \frac{z}{\eta^2 + z^2} \right| d\eta \\ &\leq \frac{4}{3\pi|z|} \int_0^{\frac{|z|}{2}} |\log(1 - e^{-2\pi\eta})| d\eta \\ &= \frac{4}{3\pi|z|} \int_0^{\frac{|z|}{2}} \log\left(\frac{1}{1 - e^{-2\pi\eta}}\right) d\eta \\ &\leq \frac{4}{3\pi|z|} \int_0^{\infty} \log\left(\frac{1}{1 - e^{-2\pi\eta}}\right) d\eta \end{aligned}$$

As $|z| \rightarrow \infty$, $|J_1(z)| \rightarrow 0$. Now let us study J_2 . At this point, we shall fix a $c > 0$ and restrict our attention to the closed right half plane $\{z = x + iy : x \geq c\}$. Suppose $\eta > \frac{|z|}{2}$. If z is in the upper quadrant, i.e. $y \geq 0$, then $|z + i\eta|^2 = x^2 + (y + \eta)^2 > x^2 + y^2 = |z|^2$, so $|z + i\eta| > |z|$. By the same argument, we have that $|z - i\eta| > |z|$ when z is in the lower quadrant. Furthermore, for all z in the half plane,

$$|z \pm i\eta| = \sqrt{x^2 + (y \pm \eta)^2} \geq \sqrt{x^2} \geq c$$

so combining results:

$$|\eta^2 + z^2| = |z - i\eta||z + i\eta| > c|z|$$

Thus:

$$\begin{aligned}
|J_2(z)| &= \left| \frac{1}{\pi} \int_{\frac{|z|}{2}}^{\infty} \log(1 - e^{2\pi\eta}) \frac{z}{\eta^2 + z^2} d\eta \right| \\
&\leq \frac{1}{\pi} \int_{\frac{|z|}{2}}^{\infty} |\log(1 - e^{2\pi\eta})| \left| \frac{z}{\eta^2 + z^2} \right| d\eta \\
&\leq \frac{1}{\pi} \int_{\frac{|z|}{2}}^{\infty} |\log(1 - e^{2\pi\eta})| \frac{|z|}{c|z|} d\eta \\
&= \frac{1}{c\pi} \int_{\frac{|z|}{2}}^{\infty} \log \left(\frac{1}{1 - e^{2\pi\eta}} \right) d\eta
\end{aligned}$$

Since $\log \left(\frac{1}{1 - e^{2\pi\eta}} \right)$ is strictly positive and its integral converges absolutely, it is clear that $J_2(z) \rightarrow 0$ as $|z| \rightarrow \infty$.

We can now compute C, C' easily using the functional equations for Γ . From (\dagger), we have:

$$\log \Gamma(z + 1) = \log z + \log \Gamma(z)$$

Plugging in our expression for Γ :

$$C' + C(z + 1) + \left(z + \frac{1}{2} \right) \log(z + 1) + J(z + 1) = \log z + C' + Cz + \left(z - \frac{1}{2} \right) \log(z) + J(z)$$

Simplifying:

$$C + \left(z + \frac{1}{2} \right) \log(z + 1) + J(z + 1) = \left(z + \frac{1}{2} \right) \log(z) + J(z)$$

Thus:

$$\implies C = \left(z + \frac{1}{2} \right) \log \left(\frac{z}{z + 1} \right) + J(z) - J(z + 1)$$

Now let $z = x \in \mathbb{R}^+$. Then:

$$\begin{aligned}
C &= \ln \left(\left(\frac{x}{x + 1} \right)^x \right) + \frac{1}{2} \ln \left(\frac{x}{x + 1} \right) + J(x) - J(x + 1) \\
&= -\ln \left(\frac{x + 1}{x} \right)^x + \frac{1}{2} \ln \left(\frac{x}{x + 1} \right) + J(x) - J(x + 1) \\
\implies C &= \lim_{x \rightarrow \infty} -\ln \left(\frac{x + 1}{x} \right)^x + \frac{1}{2} \ln \left(\frac{x}{x + 1} \right) + J(x) - J(x + 1) \\
&= -\ln(e) + \ln(0) + 0 = -1
\end{aligned}$$

So now we have:

$$\log \Gamma(z) = C' - z + \left(z - \frac{1}{2}\right) \log(z) + J(z)$$

Since (†) tells us nothing C' , we should use (★):

$$\frac{\pi}{\sin(\pi z)} = \Gamma(z)\Gamma(1-z) \implies \log \pi - \log \sin(\pi z) = \log \Gamma(z) + \log \Gamma(1-z)$$

To make things concrete, and ensure our use of the logarithm is sensible, we will set $z = \frac{1}{2} + iy$.

This is our strategy: we substitute our definition of $\log \Gamma$ into the functional equation, with z expressed as $\frac{1}{2} + iy$ where appropriate. We will eventually take the limit as $y \rightarrow \infty$ to determine the value of c . Since there are so many terms, we will study each non constant term and find a way to rewrite it as a constant plus a general error term, ϵ , that will go to 0 when $y \rightarrow \infty$. Since $J(z), J(1-z) \rightarrow 0$, for example, we can omit them from the expression, since they can be absorbed into ϵ .

$$\begin{aligned} \log(\pi) - \log(\sin(\pi z)) &= \log \Gamma(z) + \log \Gamma(1-z) \\ &= \left(C' - \frac{1}{2} - iy + iy \log\left(\frac{1}{2} + iy\right)\right) + \left(C' - \frac{1}{2} + iy - iy \log\left(\frac{1}{2} - iy\right)\right) + \epsilon \\ &= 2C' - 1 + iy \log\left(\frac{\frac{1}{2} + iy}{\frac{1}{2} - iy}\right) + \epsilon \end{aligned}$$

Furthermore:

$$\log\left(\frac{1 + \frac{1}{2iy}}{1 - \frac{1}{2iy}}\right) + i\pi = \log\left(\frac{1 + \frac{1}{2iy}}{1 - \frac{1}{2iy}}\right) + \log(e^{\pi i}) = \log\left(\frac{1 + \frac{1}{2iy}}{\frac{1}{2iy} - 1}\right) = \log\left(\frac{\frac{1}{2} + iy}{\frac{1}{2} - iy}\right)$$

The expression $\log\left(\frac{1 + \frac{1}{2iy}}{1 - \frac{1}{2iy}}\right)$ is easier to work with, since for all $y > \frac{1}{2}$, $|\frac{1}{2iy}| < 1$.

For all $|z| < 1$, we have:

$$\frac{d}{dz} \log(1+z) = \frac{1}{1+z} = \sum_{i=0}^{\infty} (-1)^i z^i \implies \log(1+z) = \sum_{i=1}^{\infty} \frac{(-1)^{i+1} z^i}{i}$$

$$\frac{d}{dz} \log(1-z) = -\frac{1}{1-z} = -\sum_{i=0}^{\infty} z^i \implies \log(1-z) = -\sum_{i=1}^{\infty} \frac{z^i}{i}$$

Thus for all sufficiently large y :

$$\begin{aligned}
y \log \left(\frac{1 + \frac{1}{2iy}}{1 - \frac{1}{2iy}} \right) &= y \left(\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n(2iy)^n} + \sum_{n=1}^{\infty} \frac{1}{(2iy)^n n} \right) \\
&= y \left(\sum_{n=1}^{\infty} \frac{(-1)^{n+1} + 1}{n(2iy)^n} \right) \\
&= -y \left(\sum_{n=1}^{\infty} \frac{2}{(2n-1)(2iy)^{2n-1}} \right) \\
&= \left(\sum_{n=1}^{\infty} (i^{-1-2n} 2^{2(1-n)} y^{2(1-n)}) \frac{1}{2n-1} \right) \\
&= -i + \left(\sum_{n=2}^{\infty} (i^{-1-2n} 2^{2(1-n)} y^{2(1-n)}) \frac{1}{2n-1} \right)
\end{aligned}$$

The summation in the final display is a power series in $\frac{1}{y}$, so it goes to 0 as $y \rightarrow \infty$, and may be absorbed into ϵ :

$$2C' - 1 + i(\pi i - i + \epsilon) = 2C' - \pi y + \epsilon$$

Now let us see what happens to the LHS when we take the limit. First, observe that:

$$\sin \left(\frac{\pi}{2} + i\pi y \right) = \cosh(\pi y)$$

so we have:

$$\log \pi - \log(\cosh(\pi y)) = 2C' - \pi y + \epsilon$$

Again, we use a Taylor series to approximate $\log(\cosh(\pi y))$.

$$\cosh(\pi y) = \frac{1 + e^{-2\pi y}}{2e^{-\pi y}} \implies \log \cosh(\pi y) = \log(1 + e^{-2\pi y}) - \log(2) + \pi y$$

For $y > 0$, $e^{-2\pi y} < 1$ so we may use the Taylor series we derived above to see:

$$\log \cosh(\pi y) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1} e^{-2\pi y n}}{n} - \log(2) + \pi y$$

The summation clearly goes to 0 as $y \rightarrow \infty$, so altogether we have:

$$\begin{aligned}
\log \pi - (\epsilon' - \log(2) + \pi(y)) &= \log 2\pi - \pi y - \epsilon' = 2C' - \pi y + \epsilon \implies C' = \log \sqrt{2\pi} \\
\implies \log \Gamma(z) &= \log \sqrt{2\pi} - z + \left(z - \frac{1}{2} \right) \log(z) + J(z)
\end{aligned}$$

$$\therefore \boxed{\Gamma(z) = \sqrt{2\pi} z^{z-\frac{1}{2}} e^{-z} e^{J(z)}}$$

This is Stirling's formula. The idea is that, since $J(z) \rightarrow 0$, we can ignore the $e^{J(z)}$ term and simply compute $\sqrt{2\pi}z^{z-1/2}e^{-z}$ to get an estimate for $\Gamma(n)$. Furthermore, restricting ourselves to the positive real axis, we can take $c = x$ and see that:

$$\begin{aligned} |J(x)| &= |J_1(x) + J_2(x)| \leq |J_1(x)| + |J_2(x)| \\ &\leq \frac{4}{3\pi x} \int_0^{\frac{x}{2}} \log\left(\frac{1}{1 - e^{-2\pi\eta}}\right) d\eta + \frac{1}{x\pi} \int_{\frac{x}{2}}^{\infty} \log\left(\frac{1}{1 - e^{-2\pi\eta}}\right) d\eta \\ &< \frac{4}{3\pi x} \int_0^{\infty} \log\left(\frac{1}{1 - e^{-2\pi\eta}}\right) d\eta \end{aligned}$$

Thus, on the positive real axis, $|J(x)|$ is bounded above by $\frac{\lambda}{x}$ for $\lambda = \frac{4}{3\pi} \int_0^{\infty} \log\left(\frac{1}{1 - e^{-2\pi\eta}}\right) d\eta$, i.e. $J(x) = \mathcal{O}\left(\frac{1}{x}\right)$, so $e^{J(x)} \rightarrow 1$ especially fast on the real axis.

Here is a table of values we get for small n :

n	$\sqrt{2\pi}n^{n-\frac{1}{2}}e^{-n}$	$\Gamma(n)$
1	0.9221...	1
2	0.9595...	1
3	1.9454...	2
4	5.8765...	6
5	23.6038..	24
6	118.346...	120

This is great! The formula we've derived does exactly what we would like, and it has the form we expected it to.

5. APPENDIX

In this appendix I will prove the following elementary result mentioned in [2]:

$$\frac{n^{n-1}}{e^{n-1}} < (n-1)! < \frac{n^n}{e^{n-1}}$$

The only tools we need are the definition of the exponential in terms of the factorial function, and the binomial theorem. For that reason, it seems appropriate to include it, since it would seem likely that one could only have to alter this proof slightly to make it work for other ‘‘Gamma-like’’ functions defined on other discrete sets (cf [3]).

Lemma 5.1. *Let $n > 1$ be an integer. Then:*

$$\frac{1}{n!} - \frac{1}{(n+1)!} > \frac{1}{n^n}$$

Proof.

$$\frac{1}{n!} - \frac{1}{(n+1)!} = \frac{n+1-1}{(n+1)!} = \frac{n}{(n+1)!} = \frac{1}{(n^2-1)(n-2)!} > \frac{1}{n^n}$$

□

Lemma 5.2. *For every positive integer k , the following inequalities hold:*

$$\left(1 + \frac{1}{k}\right)^k < e < \left(1 + \frac{1}{k}\right)^{k+1}$$

Proof. The first inequality follows almost trivially from the definition of e , and the binomial theorem:

$$\left(1 + \frac{1}{k}\right)^k = \sum_{i=0}^k \binom{k}{i} \frac{1}{k^i} = \sum_{i=0}^k \frac{k!}{i!(k-i)!} \cdot \frac{1}{k^i} = \sum_{i=0}^k \frac{k!}{(k-i)!k^i} \cdot \frac{1}{i!}$$

Now, $\frac{k!}{(k-i)!} = k(k-1)\cdots(k-i+1) < k^i$, so $\frac{k!}{(k-i)!k^i} < 1$ and thus:

$$\left(1 + \frac{1}{k}\right)^k = \sum_{i=0}^k \frac{k!}{(k-i)!k^i} \cdot \frac{1}{i!} < \sum_{i=0}^k \frac{1}{i!} < \sum_{i=0}^{\infty} \frac{1}{i!} = e$$

To prove the second inequality, we set $j = k + 1$ and prove the following equivalent statement about $j > 1$:

$$\left(1 - \frac{1}{j}\right)^j < e^{-1} = \sum_{i=0}^{\infty} \frac{(-1)^i}{i!}$$

The limit of an alternating series such as this one used to represent e^{-1} satisfies very convenient bounds:

$$\sum_{i=0}^{2\ell-1} \frac{(-1)^i}{i!} < e^{-1} < \sum_{i=0}^{2\ell} \frac{(-1)^i}{i!} \quad (\ell \in \mathbb{N})$$

That is, e^{-1} is trapped between consecutive partial sums, where the partial sum of even index is always the upper bound and the partial sum of odd index is the lower bound. Consequently, it suffices to show that $\left(1 - \frac{1}{j}\right)^j < \sum_{i=0}^{\ell} \frac{(-1)^i}{i!}$ for some odd integer ℓ .

On the one hand, $\frac{1}{k!} - \frac{1}{(k+1)!} = \frac{1}{k+1-k}(k+1)! = \frac{k}{(k+1)!}$. On the other, we have:

$$\begin{aligned} \binom{n}{k} \frac{1}{n^k} - \binom{n}{k+1} \frac{1}{n^{k+1}} &= \frac{n(n-1)\cdots(n-k+1)}{n^k \cdot k!} - \frac{n(n-1)\cdots(n-k)}{n^{k+1} \cdot (k+1)!} \\ &= \frac{n(n-1)\cdots(n-k+1)[n(k+1) - (n-k)]}{n^{k+1}(k+1)!} \\ &= \frac{n(n-1)\cdots(n-k+1)[(n+1)k]}{n^{k+1}(k+1)!} \\ &= \frac{(n^2-1)(n-2)(n-3)\cdots(n-k+1)}{n^k} \frac{k}{(k+1)!} \\ &< \frac{1}{k!} - \frac{1}{(k+1)!} \end{aligned}$$

Thus, for odd n , say $n = 2r + 1$:

$$\begin{aligned} \left(1 - \frac{1}{n}\right)^n &= \sum_{i=0}^n \binom{n}{i} \frac{(-1)^i}{n^i} \\ &= -\frac{1}{n^n} + \sum_{j=0}^r \binom{n}{2j} \frac{1}{n^{2j}} - \binom{n}{2j+1} \frac{1}{n^{2j+1}} \\ &< \sum_{j=0}^r \frac{1}{k!} - \frac{1}{(k+1)!} \\ &< e^{-1} \end{aligned}$$

For even n , say $n = 2r$ we use Lemma 7.1 to show:

$$\begin{aligned} \left(1 - \frac{1}{n}\right)^n &= \sum_{i=0}^n \binom{n}{i} \frac{(-1)^i}{n^i} = \frac{1}{n^n} + \sum_{j=0}^{r-1} \binom{n}{2j} \frac{1}{n^{2j}} - \binom{n}{2j+1} \frac{1}{n^{2j+1}} \\ &< \left(\frac{1}{n!} - \frac{1}{(n+1)!}\right) + \sum_{j=0}^{r-1} \frac{1}{k!} - \frac{1}{(k+1)!} < e^{-1} \end{aligned}$$

□

Proposition 5.3.

$$\frac{n^{n-1}}{e^{n-1}} < (n-1)! < \frac{n^n}{e^{n-1}}$$

Proof. Multiplying the inequalities we obtained in the previous lemma for $k = 1, 2, n-1$ gives a bounds for e^{n-1} :

$$\prod_{k=1}^{n-1} \left(1 + \frac{1}{k}\right)^k = \prod_{k=1}^{n-1} \left(\frac{k+1}{k}\right)^k = \prod_{k=1}^{n-1} \frac{(k+1)^k}{k^k} = \frac{\prod_{k=2}^n k^{k-1}}{\prod_{k=1}^{n-1} k^k} = \frac{n^{n-1} \prod_{k=2}^{n-1} k^{k-1}}{\prod_{k=1}^{n-1} k^k} = \frac{n^{n-1}}{(n-1)!} < e^{n-1}$$

Similarly:

$$\prod_{k=1}^{n-1} \left(1 + \frac{1}{k}\right)^{k+1} = \prod_{k=1}^{n-1} \left(\frac{k+1}{k}\right)^{k+1} = \prod_{k=1}^{n-1} \frac{(k+1)^{k+1}}{k^{k+1}} = \frac{\prod_{k=2}^n k^k}{\prod_{k=1}^{n-1} k^{k+1}} = \frac{n^n \prod_{k=2}^{n-1} k^k}{\prod_{k=1}^{n-1} k^{k+1}} = \frac{n^n}{(n-1)!} > e^{n-1}$$

So we have:

$$\frac{n^{n-1}}{(n-1)!} < e^{n-1} < \frac{n^n}{(n-1)!} \implies \boxed{\frac{n^{n-1}}{e^{n-1}} < (n-1)! < \frac{n^n}{e^{n-1}}}$$

□

REFERENCES

- [1] L.V. Ahlfors. *Complex Analysis: an Introduction to the Theory of Analytic Functions of One Complex Variable*. International Series in Pure and Applied Mathematics, 1979.
- [2] E. Artin. *The Gamma Function*, 1931.
- [3] M. Bhargava. *The Factorial Function and Generalizations*. The American Mathematical Monthly, Vol. 107, No. 9 (Nov., 2000), pp. 783-799.